Scattering Expansion for Localization in One Dimension

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We present a perturbative approach to a broad class of disordered systems in one spatial dimension. Considering a long chain of identically disordered scatterers, we expand in the reflection strength of any individual scatterer. This expansion accesses the full range of phase disorder from weak to strong. As an example application, we show analytically that in a discrete-time quantum walk, the localization length can depend non-monotonically on the strength of phase disorder (whereas expanding in weak disorder yields monotonic decrease). Returning to the general case, we obtain to all orders in the expansion a particular non-separable form for the joint probability distribution of the log-transmission and reflection phase. Furthermore, we show that for weak local reflection strength, a version of the scaling theory of localization holds: the joint distribution is determined by just three parameters.

Introduction.—Localization of waves by disorder occurs in a broad range of settings, including electron transport (in solids and mesoscopic devices), classical optics, acoustics, and Bose-Einstein condensates [1]. Progress in the general theory of localization, independent of model details or of physical realization, can then have similarly broad implications. Another setting for localization, of recent interest for its potential for quantum computing [2, 3], is the discrete-time quantum walk (DTQW), which is a quantum version of the classical random walk (see [4] for a recent review). Localization has been demonstrated in DTQWs both experimentally and theoretically (e.g., [5–11]), and it could impact quantum computing proposals even in the idealized limit of no dechoerence [12, 13].

Let us recall some general properties of disordered systems in one spatial dimension, in which case localization can be characterized by the suppression of scattering through a disordered region. The typical transmission coefficient T of a long region of length L decays exponentially: $T_{\text{typical}} \sim e^{-2L/L_{\text{loc}}}$, which defines the localization length L_{loc}. In the transfer matrix approach, rigorous theorems for random matrices demonstrate that the probability distribution $P_L(-\ln T)$ over disorder realizations is Gaussian for large L [14]. All dependence of $P_L(-\ln T)$ on L and on disorder thus reduces to two parameters (the mean and variance); this reduction is connected to ideas of universality and the renormalization group through the scaling theory of localization [15] (see also, e.g., [16, 17]). A further reduction called single-parameter scaling (SPS), in which the two parameters reduce to one by an equation relating them, was originally obtained using an assumption of phase uniformity [18], but has since been shown to hold in certain limits even without this assumption [19–21].

In this paper, we advance the general theory of localization with a perturbative approach to a general class of one-dimensional systems, and we apply our approach to DTQWs. We consider a long chain of single channel scatterers that are independently and identically disordered, and we expand in the magnitude of the reflection amplitude for any individual scatterer [22]. This expansion is particularly well-suited to study disorder that affects the phases of the reflection amplitudes but not their magnitudes, for in this case we can access the full range of disorder from weak to strong. Furthermore, by considering the expansion to all orders, we extend some of the general theory for the distribution of $-\ln T$ to the joint distribution of $-\ln T$ and of the reflection phase.

We now summarize our results in more detail. Our first main result is the expansion of the inverse localization length. We construct this expansion recursively and show that all orders depend only on local averages (that is, disorder averages over any single site). We present the first two non-vanishing orders explicitly, recovering at the leading order an equivalent formula derived by Schrader et al. [21]. In the course of our calculation, we also obtain a recursive expansion of the probability distribution of the reflection phase, finding that it is generally uniform only at the zeroth order (again extending results from [21]).

As an example application of our result, we calculate the localization length analytically as a function of phase disorder in a two-component DTQW in one dimension. We expect our calculation to apply to scattering setups [23] and beyond, and indeed we verify that our result interpolates between known results for weak and strong disorder that were calculated without reference to scattering [10]. Our expansion strictly applies when the quantum "coin" is highly biased (see below), but taking the first two non-vanishing orders yields favorable agreement with numerics even if the coin is only moderately biased. We find that the localization length can be non-monotonic in the disorder strength, i.e., there can be an amount of phase disorder beyond which further increase makes the quantum walk *less* localized (behavior seen numerically in [10]) [24].

Our second main result concerns the joint probability distribution $P_L(-\ln T, \phi)$, where ϕ is the reflection phase of the disordered region. We use an ansatz to find that for large L and to all orders in the scattering expansion, $P_L(-\ln T, \phi)$ tends to a Gaussian function (of $-\ln T$) with mean, variance, and overall scale all depending on ϕ and all calculable in the scattering expansion in terms of local averages. We further show that at the leading order in the local reflection strength (the same regime in which SPS applies to the distribution of $-\ln T$ [21, 25]), the scaling theory applies; that is, the joint distribution is determined by three parameters, which we may take to be the mean of $-\ln T$ and the mean and variance of ϕ. The latter two reach constant values for large system size.

In another manuscript [25] , we will present more applications (including the Anderson model, a quantum particle scattering on a broad class of periodic-on-average random potentials, and the "transparent mirror" effect [26] in classical optics), as well as further details of our calculations below.

Setup.—We consider a general model of scattering through a disordered region (Fig. 1). The region con-

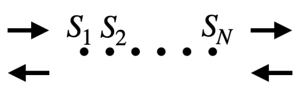

FIG. 1. Schematic of our setup.

sists of N sites labelled as $n = 1, \dots, N$, where each site n is associated with a unitary S matrix S_n parametrized as

$$S_n = \begin{pmatrix} t_n & r'_n \\ r_n & t'_n \end{pmatrix}, \tag{1}$$

where t_n and t'_n (r_n and r'_n) are the local transmission (reflection) amplitudes. We consider only the single channel case, i.e., these amplitudes are complex numbers and not matrices. We take the disorder distribution of the S matrices to be independently and identically distributed (i.i.d.) across the N sites; correlation between the entries of each individual S_n is allowed as long as every site has the same distribution.

The S matrix for the region is obtained in the usual way by multiplying transfer matrices and is parametrized as in (1), with (e.g.) $t_{1\dots N} \equiv \sqrt{T_{1\dots N}} e^{i\phi_{t_{1\dots N}}}$ and $r'_{1\dots N} \equiv \sqrt{R_{1\dots N}} e^{i\phi_{r'_{1\dots N}}}$. (We use subscripts to indicate dependence on the disorder parameters of the corresponding site or sites.) We define $s_{1\dots N} = -\ln T_{1\dots N}$ for convenience, and we write the joint probability distribution of s and $\phi_{r'}$ for the region as $P_{1\dots N}(s, \phi_{r'}) \equiv \langle \delta(s - s_{1\dots N}) \delta(\phi_{r'} - \phi_{r'_{1\dots N}}) \rangle_{1\dots N}$, where angle brackets indicate disorder averaging over the site or sites listed in the subscript. Our task is to determine properties of $P_{1\dots N}(s, \phi_{r'})$, including the localization length (which is a property of the marginal distribution of s), given the disorder distribution of the parameters of the local S matrix (1).

A basic assumption of our calculation is that localization occurs: that is, for large N the region reflection coefficient $R_{1\dots N} \approx 1$ in all disorder realizations [27]. The well-known exact recursion relations that determine $s_{1\dots N+1}$ and $\phi_{r'_{1\dots N+1}}$ from $s_{1\dots N}$, $\phi_{r'_{1\dots N}}$, r_{N+1}, and r'_{N+1} then simplify for large N to

$$s_{1\dots N+1} = s_{1\dots N} + g_{N+1}(\phi_{r'_{1\dots N}}), \tag{2a}$$

$$\phi_{r'_{1\dots N+1}} = \phi_{r'_{1\dots N}} + h_{N+1}(\phi_{r'_{1\dots N}}) \qquad (\mathrm{mod}\ 2\pi), \tag{2b}$$

where $g_n(\phi) = -\ln T_n + \ln\left(1 - r_n e^{i\phi} - r_n^* e^{-i\phi} + R_n\right)$, $h_n(\phi) = \pi - i \ln\left(\frac{1 - r_n^* e^{-i\phi}}{1 - r_n e^{i\phi}} \frac{r_n r'_n}{R_n}\right)$, $R_n = |r_n|^2 = |r'_n|^2$, and $T_n = 1 - R_n$. Eqs. (2a)-(2b) are the starting point for our analytical work, though we use the exact recursion relations in our numerical checks.

Our scattering expansion consists of rescaling $r_n \to \lambda r_n$ and $r'_n \to \lambda r'_n$ [28] in Eq. (1) (with t_n and t'_n also rescaled to maintain unitarity), then expanding in the parameter λ while simultaneously sending $N \to \infty$ in a λ-dependent way such that the system is always in the localized regime. In particular, we suppose that for any fixed $\lambda > 0$ there is some $N_0(\lambda)$ for which $R_{1\dots N} \approx 1$ for any $N \geq N_0(\lambda)$ in all disorder realizations [29], and we always work in the regime $\lambda > 0$ and $N \geq N_0(\lambda)$. Below, we suppress λ and refer informally to an expansion in $|r_n|$.

Scattering expansion of the localization length.—We start by expressing the localization length in terms of the limiting form $p_{1\dots\infty}(\phi_{r'}) \equiv \lim_{N\to\infty} \int_0^\infty ds\ P_{1\dots N}(s, \phi_{r'})$ of the marginal distribution of the reflection phase. From Eq. (2a), we see that for sufficiently large N, $\langle s_{1\dots N} \rangle_{1\dots N}$ increases by the same constant amount (which by definition is $2/L_{\mathrm{loc}}$ [30]) each time N is increased by one: $\langle s_{1\dots N+1} \rangle_{1\dots N+1} - \langle s_{1\dots N} \rangle_{1\dots N} = \int_{-\pi}^{\pi} d\phi\ p_{1\dots\infty}(\phi) \langle g_{N+1}(\phi) \rangle_{N+1} = 2/L_{\mathrm{loc}}$ [31]. There is in fact no dependence on $N+1$ seeing as the (i.i.d.) disorder average can be done over any site n [32]. Converting to Fourier space yields a series expression for the inverse localization length in terms of the Fourier components $p_{1\dots\infty,\ell} \equiv \int_{-\pi}^{\pi} \frac{d\phi_{r'}}{2\pi} e^{-i\ell\phi_{r'}} p_{1\dots\infty}(\phi_{r'})$ and the moments of r_n [25]:

$$\frac{2}{L_{\mathrm{loc}}} = \langle -\ln T_n \rangle_n - 4\pi \mathrm{Re} \left[\sum_{\ell=1}^{\infty} \frac{1}{\ell} p_{1\dots\infty,-\ell} \langle r_n^\ell \rangle_n \right]. \tag{3}$$

Eq. (3) recovers the uniform phase formula $2/L_{\mathrm{loc}} = \langle -\ln T_n \rangle_n$ [18] in two non-exclusive special cases: (i) the local reflection phase is uniformly distributed independently of the local reflection coefficient (for then $\langle r_n^\ell \rangle_n = 0$ for $\ell > 0$), or (ii) the reflection phase distribution of the region is uniform. The difficulty of applying Eq. (3), in the case that (i) does not hold, is that it has been shown in many examples that the reflection phase distribution can be non-uniform, and in general the distribution is only known numerically (although Schrader et al. [21] calculated $p_{1\ldots\infty,\ell=-1}$ in an equivalent form) [33].

The key advance that we make is to apply the scattering expansion to $p_{1\ldots\infty}(\phi_{r'})$, showing that it may written as a recursively-defined series involving only *local* averages. Furthermore, we show that $p_{1\ldots\infty,\ell} = O(|r_n|^{|\ell|})$, which implies that only finitely many terms in the sum in Eq. (3) are needed to obtain the scattering expansion of the inverse localization length to any fixed order. These results rely on the disorder distribution being "reasonable" and the particular model parameters chosen being "generic;" our precise assumptions are that localization occurs and that the inequality $\left\langle(-r_n r_n'/R_n)^\ell\right\rangle_n \neq 1$ holds for all integers $\ell \neq 0$ [34].

We focus here on the results of this calculation; details will be presented elsewhere [25]. It is convenient to define $v_n = r_n r_n'/R_n$, $\alpha_\ell = 1/[1-\langle(-v_n)^\ell\rangle_n]$, and several constants determined by local averages (we use a superscript to indicate the order of a given constant in the scattering expansion): $\gamma^{(1)} = \alpha_1\langle r_n'\rangle_n$, $\gamma^{(2)} = \alpha_2\langle r_n'(r_n' - 2\gamma^{(1)}v_n)\rangle_n$, $\gamma_1^{(3)} = \alpha_1\langle r_n(\gamma^{(1)}r_n' - \gamma^{(2)}v_n)\rangle_n$, and $\gamma_3^{(3)} = \alpha_3\langle r_n'({r_n'}^2 - 3\gamma^{(1)}r_n'v_n + 3\gamma^{(2)}v_n^2)\rangle_n$. Then we have $2\pi p_{1\ldots\infty}(\phi_{r'}) = 1 + 2\mathrm{Re}[(\gamma^{(1)}+\gamma_1^{(3)})e^{-i\phi_{r'}} + \gamma^{(2)}e^{-2i\phi_{r'}} + \gamma_3^{(3)}e^{-3i\phi_{r'}}] + O(|r_n|^4)$ and our main result for the localization length:

$$\frac{2}{L_{\mathrm{loc}}} = \langle R_n\rangle_n - 2\mathrm{Re}\left[\frac{\langle r_n\rangle_n\langle r_n'\rangle_n}{1+\langle r_n r_n'/R_n\rangle_n}\right] + \frac{1}{2}\langle R_n^2\rangle_n - 2\mathrm{Re}\Bigg[\alpha_1^2\langle r_n\rangle_n\langle r_n'\rangle_n\left(\langle r_n r_n'\rangle_n + 2\alpha_2\langle r_n v_n\rangle_n\langle r_n' v_n\rangle_n\right) + \alpha_2\left(\frac{1}{2}\langle r_n^2\rangle_n\langle r_n'^2\rangle_n - \alpha_1\langle r_n\rangle\langle r_n'^2\rangle_n\langle r_n v_n\rangle_n - \alpha_1\langle r_n'\rangle_n\langle r_n^2\rangle_n\langle r_n' v_n\rangle_n\right)\Bigg] + O(|r_n|^6). \quad (4)$$

The first two terms in Eq. (4) are the leading order contribution (second order in $|r_n|$) and were found in an equivalent form by Schrader et al. in [21]. The remaining terms are fourth order, and indeed all odd orders vanish by symmetry [25]. The terms whose real parts are taken are the contributions from the non-uniformity of the reflection phase distribution. We emphasize that these non-uniform phase contributions are parametrically of the same order as the uniform phase contributions ($\langle -\ln T_n\rangle_n = \langle R_n\rangle_n + \frac{1}{2}\langle R_n^2\rangle_n + \ldots$); in particular, deviations from phase uniformity generally affect the inverse localization length even at leading order [21, 35, 36].

Application to quantum walks.—We next apply the general result (4) to a single-step, two-component DTQW in one dimension [10]. The setup is an infinite chain with site index n and an internal "spin" degree of freedom ($\uparrow$ or $\downarrow$). The unitary operator $\hat{U}$ that implements a single time step is $\hat{U} = \sum_n (|n+1,\uparrow\rangle\langle n,\uparrow| + |n-1,\downarrow\rangle\langle n,\downarrow|)\hat{U}_{\mathrm{coin}}$, where the "coin" operator is $\hat{U}_{\mathrm{coin}} = \sum_n |n\rangle\langle n| \otimes U_{\mathrm{coin},n}$ and $U_{\mathrm{coin},n}$ is a general 2-by-2 unitary matrix (acting on the spin degree of freedom at site n) parametrized as [10]

$$U_{\mathrm{coin},n} = e^{i\varphi_n}\begin{pmatrix} e^{i\varphi_{1,n}}\cos\theta_n & e^{i\varphi_{2,n}}\sin\theta_n \\ -e^{-i\varphi_{2,n}}\sin\theta_n & e^{-i\varphi_{1,n}}\cos\theta_n\end{pmatrix}. \quad (5)$$

We take the parameters $\mathbf{D}_n \equiv (\theta_n, \varphi_n, \varphi_{1,n}, \varphi_{2,n})$ to be i.i.d. across the sites $n = 1,\ldots,N$ (note that the components of $\mathbf{D}_n$ may be correlated with each other), defining a disordered region.

The S matrix of the region describes solutions of the stationary state equation $\hat{U}|\Psi\rangle = e^{-i\omega}|\Psi\rangle$ (with quasienergy ω). There are in fact many possible scattering problems, corresponding to different choices for site-independent values to be assigned to $\mathbf{D}_n$ in the non-disordered regions (the sites $n<1$ and $n>N$). It may be shown that all choices result in a problem of the form we have been considering (i.e., there is some S matrix $\mathcal{S}_n$ that depends only on $\mathbf{D}_n$ and ω) and that the probability distribution of the transmission coefficient in the localized regime is the same in all cases [25]. We consider the simplest case of setting $\mathbf{D}_n = \mathbf{0}$ in the non-disordered regions, which results in $\mathcal{S}_n = e^{i\omega}U_{\mathrm{coin},n}$. Comparing to Eq. (5), we see that the local reflection amplitudes are $r_n = -e^{i(\omega+\varphi_n-\varphi_{2,n})}\sin\theta_n$ and $r_n' = e^{i(\omega+\varphi_n+\varphi_{2,n})}\sin\theta_n$. Then Eq. (4) yields the inverse localization length for small $\sin\theta_n$, up to an error of order $|r_n|^6 = \sin^6\theta_n$, with arbitrary phase disorder. In particular, the joint distribution of $\mathbf{D}_n$ is arbitrary as long as $\sin\theta_n$ is small.

Specializing to the case of φ_n uniformly distributed in $[-W,W]$, with $\varphi_{1,n} = \varphi_{2,n} = 0$ and $\theta_n \equiv \theta$, we obtain the inverse localization length for small $\sin\theta$ and arbitrary phase disorder strength W. We have verified that our result agrees with the calculation of Vakulchyk et al. [10], in which θ is arbitrary and W is either small (yielding $2/L_{\mathrm{loc}} \sim W^2$) or equal to π (in which case the uniform phase formula holds). Our result thus interpolates between the known limits of weak and strong phase disorder and analytically demonstrates non-monotonicity in disorder strength [37]. We have verified our result with numerics in the regime of small $\sin\theta$ [25]; furthermore, in Fig. 2 we show that the agreement with numerics is favorable even if $\sin\theta$ is not particularly small.

Joint probability distribution.—Returning to the general case, we now summarize the results of applying the

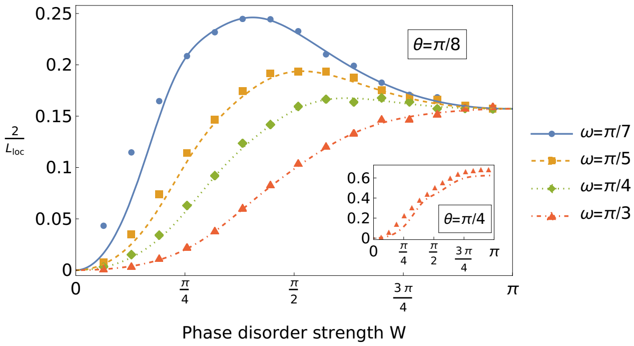

FIG. 2. The inverse localization length ($2/L_{\rm loc}$) vs. the strength of phase disorder (W) in the variable φ_n in the DTQW. We compare our theoretical result (4) (lines) with numerics (points) for a moderately biased coin (main plot) and for an unbiased coin (inset).

scattering expansion to the joint probability distribution $P_{1\ldots N}(s,\phi_{r'})$ [25]. We find that for large N this distribution takes a Gaussian form defined as follows. There is a constant c and two functions $\hat{s}(\phi_{r'})$, $\eta(\phi_{r'})$ for which we have

$$P_{1\ldots N}(s,\phi_{r'}) = p_{1\ldots\infty}(\phi_{r'}) \times \frac{1}{\sqrt{2\pi\sigma(N,\phi_{r'})^2}} e^{-\frac{1}{2}[s-\frac{2N}{L_{\rm loc}}-\hat{s}(\phi_{r'})]^2/\sigma(N,\phi_{r'})^2}, \quad (6)$$

where the phase-dependent variance $\sigma(N,\phi_{r'})^2$ scales linearly with N with a sub-leading, phase-dependent correction: $\sigma(N,\phi_{r'})^2 = 2[cN+\eta(\phi_{r'})]$. The constant c is related to the variance $\sigma(N)^2$ of the marginal distribution of s by $\sigma(N)^2 = 2cN + O(N^0)$. We can calculate the quantities c, $\hat{s}(\phi_{r'})$, and $\eta(\phi_{r'})$ order-by-order in the scattering expansion in terms of local averages [except that the functions $\hat{s}(\phi_{r'})$ and $\eta(\phi_{r'})$ each have an undetermined, $\phi_{r'}$-independent additive constant], and in particular we have obtained $c = 2/L_{\rm loc} + O(|r_n|^4)$ (i.e., SPS to one more order than shown in [21]), $\hat{s}(\phi) = 2{\rm Re}\{\gamma^{(1)}e^{-i\phi} + [\frac{3}{2}\gamma^{(2)} - (\gamma^{(1)})^2]e^{-2i\phi}\} + O(|r_n|^3) + \text{const.}$, and $\eta(\phi) = {\rm Re}\{[\gamma^{(2)} - (\gamma^{(1)})^2]e^{-2i\phi}]\} + O(|r_n|^3) + \text{const.}$

We now explain briefly how we arrive at Eq. (6). From Eqs. (2a)-(2b), it is straightforward to show that the joint probability distribution satisfies a recursion relation of the form $P_{1\ldots N+1}(s,\phi_{r'}) = \mathcal{F}[s,\phi_{r'};\{P_{1\ldots N}\}]$, where $\mathcal{F}$ is a linear functional in its last argument. We take Eq. (6) as an ansatz and require $\mathcal{F}[s,\phi_{r'};\{P_{1\ldots N}\}] = P_{1\ldots N+1}(s,\phi_{r'}) + O(1/N^2)$ for large N; this requirement fixes c, $\hat{s}(\phi_{r'})$, and $\eta(\phi_{r'})$ to all orders in the scattering expansion [except for the constant offsets of $\hat{s}(\phi_{r'})$ and $\eta(\phi_{r'})$]. Since the ansatz itself is $O(1/\sqrt{N})$, we can expect that (6) is the leading term in an expansion in $1/\sqrt{N}$ of the exact answer.

The correlation between s and $\phi_{r'}$ in (6) is a finite size effect, as we now explain. We write the average of s as $\langle s\rangle = 2N/L_{\rm loc} + O(N^0)$, and we consider how accurate $\langle s\rangle$ is as an estimate of the conditional average of s with fixed $\phi_{r'}$ in (6). The phase-dependent variation of the mean introduces a relative error of order $\hat{s}(\phi_{r'})/\langle s\rangle \sim 1/N$, while the finite variance introduces a relative error of order $\sigma(N,\phi_{r'})/\langle s\rangle = cL_{\rm loc}/\sqrt{N} + O(N^{-3/2})$, where the $N^{-3/2}$ term contains the contribution of the function $\eta(\phi_{r'})$. Prior work has found the joint probability distribution to factorize into a transmission coefficient part times a phase part [38, 39], in apparent contradiction to our Eq. (6); this suggests that the prior work only accounted for the $1/\sqrt{N}$ term in the above discussion and neglected the $1/N$ and $N^{-3/2}$ terms that contain the correlations between s and $\phi_{r'}$.

We next show that the scaling theory applies to the joint distribution in the regime of weak local reflection strength. Here we ignore $\eta(\phi_{r'})$ (whose effect is subleading for large N, as we have shown above) and expand the remaining terms of (6) to leading order in the scattering expansion. A single parameter then determines $2/L_{\rm loc}$ and c, seeing as $2/L_{\rm loc} = c$ at leading order [21]. Furthermore, the phase distribution up to first order is determined entirely by two parameters: the real and imaginary parts of $\gamma^{(1)}$, or by a simple change of variables, the mean and variance of $\phi_{r'}$. The key relation that implies that these three parameters suffice to determine the joint distribution is that the first order part of the phase-dependent mean turns out to be essentially same function as the first order part of the phase distribution:

$$\hat{s}(\phi_{r'}) = 2\pi p_{1\ldots\infty}(\phi_{r'}) + O(|r_n|^2) + \text{const.}, \quad (7)$$

where the constant on the right-hand side is independent of $\phi_{r'}$.

Conclusion.—In a general problem of single channel scattering through an i.i.d. disordered chain, we developed a systematic expansion in the local reflection strength, which we call the scattering expansion. We calculated the inverse localization length to the first two non-vanishing orders in this expansion, using an explicit expansion of the (generally non-uniform) reflection phase distribution. We applied our result to calculate the localization length in a two-component DTQW with a biased coin parameter and arbitrary phase disorder, and we thus showed analytically that the localization length can depend non-monotonically on the strength of phase disorder.

Returning to the general problem, we summarized the results of applying the scattering expansion to the joint probability distribution of the log-transmission and reflection phase: first, we found the general form of the joint distribution to all orders in the scattering expansion, and second, we showed that when the local reflection strength is weak, the joint distribution is determined by three parameters.

Our methods might extend to the quasi-one-dimensional case (i.e., multi-channel S matrices).

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[22] In the context of light propagation in disordered metamaterials, such an expansion was done in [40] (see [41] for a review); however, this work has an uncontrolled aspect (c.f. the comment below Eq. 2.19 in [41]) that we do not think is present in our work. As we comment on in the main body of this manuscript, some of the leading order results of our expansion were obtained in an equivalent form by Schrader et al. [21] (see also [42]); some calculation is required to show the equivalence [25].
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[24] In other contexts it has been shown that transmission through a disordered region can be non-monotonic in disorder strength when the incident wave is in an energy gap of the non-disordered system (see [43] and references therein); this is not the case we consider.
[25] A. B. Culver, P. Sathe, and R. Roy, (unpublished).
[26] M. V. Berry and S. Klein, Transparent mirrors: Rays, waves and localization, European Journal of Physics **18**, 222 (1997).
[27] Whenever we require a property to hold for all disorder realizations, we expect that it would suffice for that property to hold for almost all realizations (i.e., all except a set of measure zero).
[28] The natural small parameter λ in a particular problem may be such that $|r_n|$ starts at linear order in λ but also has higher-order corrections. Our results are easily modified to accommodate this more general case.
[29] A similar point is mentioned in [17] (footnote 24).
[30] Except in the Introduction section, we take the unit of

distance to be the spacing between adjacent scatterers, so that $\langle -\ln T_{1\ldots N}\rangle_{1\ldots N} \sim 2N/L_{\mathrm{loc}}$. If the scatterers are, e.g., potentials separated by i.i.d. disordered distances a_n, then the unit of distance becomes $\langle a_n\rangle_n$.

[31] An equivalent expression was obtained by Lambert and Thorpe in Refs. [35, 44].

[32] Whenever we write a disorder average over n, it is understood that n is an arbitrary site.

[33] The reflection phase distribution has been studied in a variety of particular models, both in scattering and in real-space approaches (see [39] and references therein, and also, e.g., [44–51]).

[34] This inequality fails for, e.g., the Anderson model with momentum $k = (p/q)\pi$ for some integers p, q; these momenta are known to be anomalous [50, 52]. Also, a disorder distribution equal to an appropriately chosen delta function (i.e., no disorder) can violate the inequality. However, the inequality alone is not sufficient; we have found an example in which numerics indicate that at special parameter values, localization does not occur (and hence our results do not apply), even though the inequality holds [25].

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[37] $\frac{2}{L_{\mathrm{loc}}} = \left(1 + \frac{2\,\mathrm{sinc}^2 W[\cos(2w) - \mathrm{sinc}(2W)]}{1 - 2\cos(2w)\,\mathrm{sinc}(2W) + \mathrm{sinc}^2(2W)}\right)\sin^2\theta$ is the explicit result at leading order and exhibits the non-monotonicity in W (for ω within a certain range).

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